Cor: There is a curve of deg d passing through any given $d(d+3)/2$ points

the intersection of any n hyperplanes of $\mathbb{P}^n$ is not empty.

Lemma: $T: \mathbb{P}^2 \to \mathbb{P}^2$ proj. change of coordinates. Then

$$\begin{aligned} \mathbb{P}^{d(d+3)/2} &\longrightarrow \mathbb{P}^{d(d+3)/2} \\ F &\longmapsto F^T \end{aligned}$$

is also a proj. change of coordinates.

Pf: 1) $P \in F_{[a_1:\cdots:a_N]} = \sum_i a_i M_i \iff \sum_i a_i M_i(P) = 0 \Rightarrow \checkmark$

2). T = linear $\Rightarrow T(M_1 \cdots M_N) = (M_1, \cdots, M_N) A \leftarrow GL_N(k)$

$$F^T = (M_1^T, \cdots, M_N^T)\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = (M_1, \cdots, M_N) \underbrace{A\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}}_{\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}}$$

$$\mathbb{P}^{N-1} \longrightarrow \mathbb{P}^{N-1}$$

$$F = [a_1:\cdots:a_N] \longmapsto [b_1:\cdots:b_N] = F^T \qquad \Rightarrow \checkmark$$

Lem: $\forall P \in \mathbb{P}^2$, $r \leq d+1$

$$\{ F : \text{curve of deg } d \mid m_P(F) \geq r \}$$

forms a linear subvariety of $\dim = \dfrac{d(d+3)}{2} - \dfrac{r(r+1)}{2}$

Pf: by coordinate change, we may assume $P = [0:0:1]$

$$F = \sum F_i(X,Y) \cdot Z^{d-i} \qquad F_i = \text{form of deg } i.$$

$m_p(F) \geq r \Leftrightarrow F_0 = F_1 = \cdots = F_{r-1} = 0$

$\Leftrightarrow$ coeff. of $x^i y^j z^k$ with $i+j<r$ are zero

$\frac{r(r+1)}{2} = 1+2+\cdots+r.$

Thm $P_1, \cdots, P_n \in \mathbb{P}^2$ distinct. $r_1, \cdots, r_n \geq 0$.

1) $V(d; r_1P_1, \cdots, r_nP_n) = \{F : \text{curve of deg } d \mid m_{P_i}(F) \geq r_i\}$

forms a linear subvariety of $\mathbb{P}^{d(d+3)/2}$ of

$$\dim \geq \frac{d(d+3)}{2} - \sum \frac{r_i(r_i+1)}{2}$$

2) if $\sum r_i \leq d+1$, then

$$\dim = \frac{d(d+3)}{2} - \sum \frac{r_i(r_i+1)}{2}$$

Pf: 1) $V(d; r_1P_1, \cdots, r_nP_n) = \bigcap_{i=1}^{n} V(d, r_iP_i)$

$$\Rightarrow \dim \geq \left(\frac{(d+1)(d+2)}{2} - \sum_i \frac{r_i(r_i+1)}{2} \right) - 1$$

2). induction on $m = (\sum r_i) - 1 \leq d$.

WMA : $m > 1$ & $d > 1$.

Case : $r_i = 1$ $(\forall i)$. $V_i := V(d; P_1, \cdots, P_i)$

ONTS : $V_n \neq V_{n-1}$

$\exists$ line L_i passing P_i but not P_j $(\forall j \neq i)$

$\exists$ line L_0 not passing any P_i

$F := L_1 \cdots L_{n-1} L_0^{d-n+1} \in V_{n-1} \setminus V_n$

Case $r_i > 1$ for some i. (WMA $i=1$. i.e. $r_1 > 1$)
$P_1 = [0:0:1]$

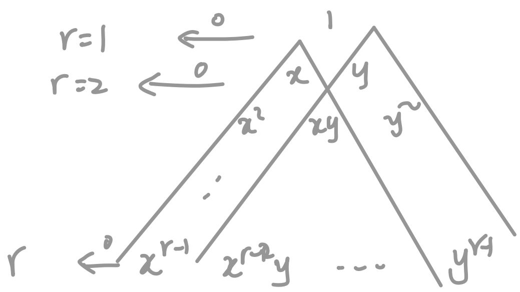
$$V_0 = V(d;\ (r_1-1)P_1,\ r_2P_2, \cdots, r_nP_n)$$

$\forall F \in V_0$ $F_* = \sum_{i=0}^{r-1} a_i x^i y^{r-1-i}$ + higher terms

$$V_i := \{ F \in V_0 \mid a_j = 0 \ \forall j < i \}$$

Then $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_r = V(d, r_1P_1, \cdots, r_nP_n)$

ONTS: $V_i \neq V_{i+1}$



$W_0 := V(d-1;\ (r_1-2)P_1,\ r_2P_2, \cdots; r_nP_n)$ inductively

$$W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_{r-1} = V(d-1;\ (r_1-1)P_1, r_2P_2, \cdots; r_nP_n)$$

$\forall F \in W_i \setminus W_{i+1}$ $\Rightarrow$ $YF_i \in V_i \setminus V_{i+1}$

$XF_i \in V_{i+1} \setminus V_{i+2}$

$\Rightarrow V_i \neq V_{i+1} \quad \forall i.$